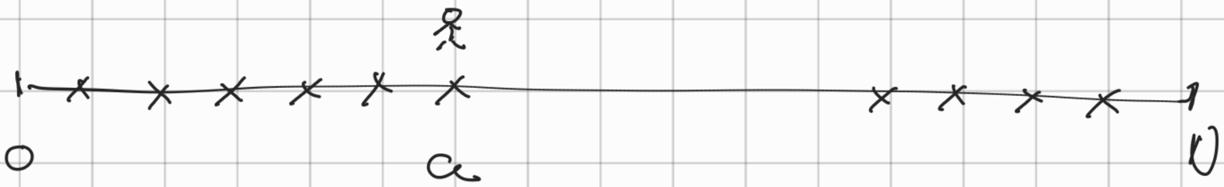
Math 3235 Probability Theory

4/20/23




Flip a coins with prop $p$ of H.

If H $\longrightarrow$ $a+1$

T $\longrightarrow$ $a-1$

$v(a)$ prob. The walk ends at N.

$1-v(a)$ prob The walk ends at 0

$v(a) = \mathbb{P}(\text{walk end at } N \mid \text{it started at } a)$

$p$ $a+1$

$1-p$ $a-1$

$$v(a) = \mathbb{P}(H)\,\mathbb{P}(\text{ends at } N \mid \text{starts at } a \;\&\; H)$$

$$+\, \mathbb{P}(T)\,\mathbb{P}(\text{ends at } N \mid \text{starts at } a \;\&\; T)$$

$$\mathbb{P}(\text{ends at } N \mid \text{starts at } a+1)$$

$$v(a) = p\, v(a+1) + q\, v(a-1)$$

$$v(N) = 1$$

$$v(0) = 0$$

$$p + q = 1$$

$$v(a) = 1 \qquad \forall a$$

$$v(a) = x^{a}$$

$$x^{a} = p x^{a+1} + q x^{a-1}$$

$$1 = p x + q x^{-1}$$

$$x = 1$$

$$x = \frac{q}{p}$$

$$p x^2 - x + q = 0$$

$$W(a) = p\, W(a+1) + q\, W(a-1)$$

$$W(a) = 1$$

$$W(a) = \left(\frac{q}{p}\right)^a$$

$$w(a) = c_1\, 1 + c_2 \left(\frac{q}{p}\right)^a$$

is a solution!

$$C_1 + C_2 = 0$$

$$C_1 + C_2 \left(\frac{q}{p}\right)^N = 1$$

$$v(a) = \frac{(q/p)^a - 1}{(q/p)^N - 1}$$

$p x^2 - 1 + q = 0$ sol. $1$ $\frac{q}{p}$

$W(a) = 1$

$W(a) = a$

if $p = q = \frac{1}{2}$

$$W(a) = \frac{1}{2} W(a+1) + \frac{1}{2} W(a-1)$$

$W(a) = a$

$W(a) = C_1 + C_2 a$

$$\left.\begin{array}{l} C_1 + C_2 \cdot 0 = 0 \\ C_1 + C_2 \cdot N = 1 \end{array}\right\| \Longrightarrow \begin{array}{l} C_1 = 0 \\ C_2 = \frac{1}{N} \end{array}$$

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$$b_0 v(a) + b_1 v(a+1) + b_2 v(a+2) \ldots b_k v(a+k) = 0$$

$$v(a) = p\, v(a+1) + q\, v(a-1)$$

$$\Downarrow$$

$$p\, v(a+2) - v(a+1) + q\, v(a) = 0$$

Guess $v(a) = x^a$

gives you a polynomial of degree $K$. with roots $x_1 \dots x_k$

$$v(a) = c_1 x_1^a + c_2 x_2^a + \dots c_k x_k^a$$

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$p = q$

$$v(a) = \frac{a}{N} \qquad 1 - v(a) = \frac{N-a}{N}$$

Game ends at $N$ I won $N-a$

Game ends at $0$ I won $-a$

$$\mathbb{E}(\text{win}) = \frac{a}{N}(N-a) - \frac{(N-a)a}{N} = 0$$

$x$ at time $t$

$x+1$ prob $\frac{1}{2}$

$t+1$

$x-1$ prob $\frac{1}{2}$

State of the system

$\Pi_t(x)$ prob of being at $x$ at time $t$.

$$\begin{cases} \Pi_{t+1}(x) = \frac{1}{2}\Pi_t(x+1) + \frac{1}{2}\Pi_t(x-1) \\ \Pi_0(x) \quad \text{given} \end{cases}$$

$$\Pi_0(x) = \begin{cases} 1 & x = N/2 \\ 0 & \text{otherwise} \end{cases}$$

$$\Pi_{t+1}(x) - \Pi_t(x) = \frac{1}{2}\Big(\Pi_t(x+1) - 2\Pi_t(x) + \Pi_t(x-1)\Big)$$

I'm looking for a steady state:

$$\bar{\pi}(a)$$

$$\bar{\pi}(a+1) - 2\bar{\pi}(a) + \bar{\pi}(a-1) = 0$$

$$\bar{\pi}(N) = \bar{\pi}(0)$$

$$\bar{\pi}(a) = \frac{1}{N}$$

$$\pi_0(a)$$

$$\sum_a \left| \pi_t(a) - \bar{\pi}(a) \right| \le e^{-\lambda t} C$$

$$\lambda \simeq \frac{1}{N}$$